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Disentangling the effect of jumps on systematic risk using a new estimator of integrated co-volatility

Kent Wang^a, Junwei Liu^a, Zhi Liu^{b,*}^a The Wangyanan Institute for Studies in Economics, Xiamen University, China^b Department of Mathematics, University of Macau, Macau

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ABSTRACT

We propose a new threshold-pre-averaging realized estimator for the integrated co-volatility of two assets using non-synchronous observations with the simultaneous presence of microstructure noise and jumps. We derive a noise-robust Hayashi–Yoshida estimator that allows for very general structure of jumps in the underlying process. Based on the new estimator, different aspects and components of co-volatility are compared to examine the effect of jumps on systematic risk using tick-by-tick data from the Chinese stock market during 2009–2011. We find controlling for jumps contributes significantly to the beta estimation and common jumps mostly dominate the jump's effect, but there is also evidence that idiosyncratic jumps may lead to significant deviation. We also find that not controlling for noise and jumps in previous realized beta estimations tend to considerably underestimate the systematic risk.

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1. Introduction

In the one factor capital asset pricing model (CAPM), systematic risk, measured by beta, is determined by covariance with the market (Sharpe, 1963; Lintner, 1965). The betas are not directly observable. The traditional way of circumventing this problem and estimating betas has relied on rolling linear regression, typically based on 5 years of monthly data.¹ The recent advent of readily available high frequency data has spurred a revived interest into alternative ways to more accurately estimate betas.² Compared to traditional parametric methods, a non-parametric approach using high frequency data is useful in that it trivializes calculation and

avoids many distortive assumptions necessary for parametric modeling.³ Studies show that the use of high frequency data results in statistically superior beta estimates relative to the traditional regression based procedures. Unlike the constant period-by-period beta from the CAPM, the realized beta model allows continuous evaluation in the beta estimation and thus provides an estimator for measurement of time varying systematic risk.

However, estimating realized variance or covariance measures from high frequency data will inevitably face problems such as microstructure noise and non-synchronous trading. Set against that background, numerous studies have been proposed to determine how to mitigate the above two problems. Among the latest endeavors, Christensen et al. (2010) propose an estimator for the realized covariance while controlling for possible microstructure noise and non-synchronous trading distortions. They develop a pre-averaged version of the Hayashi and Yoshida (2005) estimator

* Corresponding author. Tel.: +853 8397 8544; fax: +853 2883 8314.

E-mail addresses: kentwang@xmu.edu.cn (K. Wang), wjl.wise@gmail.com (J. Liu), liuzhi@umac.mo (Z. Liu).¹ See, e.g., the classical works by Fama and MacBeth (1973).² In particular, Andersen et al. (2005) and Barndorff-Nielsen and Shephard (2004) among others, have all explored new procedures for measuring and forecasting interval betas based on realized variation measures constructed from the summation of squares and cross-products of higher frequency within period returns.³ An intuitive approach has been made by Merton (1980), while others, including but not limited to Andersen et al. (2003) and Barndorff-Nielsen and Shephard (2006), have worked on the development of rigorous non-parametric estimators.

(HY hereafter) that can be directly implemented on the raw noisy and non-synchronous observations, without any prior alignment of prices. The Christensen et al. (2010) work, however, has not considered jumps, of which the existence and importance has been generally acknowledged in the high frequency literature. It is clear that jumps play a significant role in high-frequency based realized variation estimation. A large body of literature has evolved to show both theoretically and empirically that jumps explain many of the dynamic features of stylized facts documented in asset prices.⁴ The presence of jump variations in both individual assets and the aggregate market will affect co-volatility estimation and consequently the measurement of realized beta and systematic risk. The missing link of Christensen et al. (2010) for jumps thus provides important insights to improve the estimation of the integrated co-volatility, which is what we propose to do in the current study.

The new estimator we propose is obtained in the presence of microstructure noise, non-synchronous trading, and possible jumps from individual assets or the market. The timing of jump occurrence in the two assets is an important issue from such a perspective. Jacod and Todorov (2009) develop the idea and formal tests for “common jumps” (jumps occurring at the same time) and “disjoint jumps” (not occur at the same time but within the same day) for a bivariate process that is observed on a finite time interval at discrete times. This paper follows the Jacod and Todorov (2009) definitions in discussing of our decomposition of jumps using the proposed co-volatility estimator.

By incorporating jumps as an additional factor, the new estimator, through a realistic cojumps threshold setting that accounts for the interaction between the two assets, can possibly disentangle the effect of idiosyncratic jumps and cojumps⁵ on systematic risk. Our approach here, however, is different from that of Todorov and Bollerslev (2010). Todorov and Bollerslev (2010) provide a new theoretical framework for disentangling and estimating the sensitivity towards systematic diffusive and jump risk in the context of factor models. They focus on the decomposition of systematic risk by recognizing the jump occurrence at aggregate market level and show that diffusive and jump betas with respect to aggregate market portfolio differ substantially. Their work only considers one process for estimating beta risk while ours considers two processes and allows for more general jump settings at both aggregate market and individual asset levels. In addition, Todorov and Bollerslev (2010) do not consider microstructure noise effects and hence their method cannot be used for ultra-high frequency data as our approach can. Given the recent development in empirical studies to show the importance of both idiosyncratic jumps and systematic jumps,⁶ our approach is hence an alternative to the Todorov and Bollerslev (2010) framework for estimating realized systematic risk, with less restrictions imposed from our approach (i.e. our methodology is not based on factor models and can be generalized to general integrated co-volatilities).

Our simulation results suggest that the newly proposed estimator can produce much more accurate and robust estimations for the integrated co-volatility and it can disentangle relative contributions of the different aspects to integrated co-volatility. Con-

trasting the three alternatives, i.e. estimate without noise and jump accounted for, with noise correction only, and with jump correction only, the new estimator we propose accounting for both noise and jump produces the least bias. In the empirical illustration, we apply the new realized covariance estimator to tick-by-tick data from a major Chinese stock market index and four blue chip Chinese stocks⁷ representing major Chinese business industries and calculate the realized betas for the four stocks following the definition of Andersen et al. (2006).

In the empirical illustration we obtain several important findings: First, compared to noise correction and other aspects, controlling for jumps affects the estimation of co-volatility significantly. The proportion for the total jumps’ “contribution” to co-volatility estimation is shown to be approximately 30%, suggesting that our proposed estimator has great importance in correcting biases in integrated co-volatility estimation. Second, through our decomposition approach, we show that common jumps play a more prominent role in affecting the co-volatility estimation than idiosyncratic jumps. We also find that it is possible for the idiosyncratic jumps to dominate the jumps’ effect, suggesting that both common jumps and idiosyncratic jumps can have significant effect on the estimation of integrated co-volatility. This is consistent with positions taken by some recent publications such as Lee (2012) who explicitly show that distinguishing idiosyncratic jumps from systematic jumps is both possible and important. Third, in comparing effects of different aspects on realized beta estimation, we find that jumps likely cause upward biases for beta estimation while microstructure noises usually lead to downward biases. The combined effects of the two aspects are usually nonlinear and lead to underestimation of systematic risk. By controlling for both noise and jump effects, based on our sample, the proposed new estimator indicated that estimated systematic risk should be 50% higher than otherwise neither effect is accounted for.

We organize the rest of the paper as follows: Section 2 provides theoretical methodologies on the new estimator for co-volatility accounting for jumps; Section 3 discusses the simulation results for the new estimator; Section 4 presents the empirical illustration in the Chinese market, disentangling the effect of jumps on estimation of the co-volatility and systematic risk; and we summarize and conclude in Section 5. And all the technical proofs are postponed to Appendix A.

2. Methodology

2.1. The basic model setting

Suppose that we have a d -dimensional underlying log price process of assets, \mathbf{X}_t . A standard no-arbitrage condition suggests that security prices must be semi-martingales.⁸ Consequently, a widely used model for \mathbf{X}_t is the following semi-martingale:

$$\mathbf{X}_t = \mathbf{X}_t^C + \mathbf{X}_t^D, \quad (1)$$

where \mathbf{X}_t^C and \mathbf{X}_t^D are, respectively, the continuous and discontinuous components whose explicit forms are the following:

$$\mathbf{X}_t^C = \mathbf{X}_0 + \int_0^t \mathbf{b}_t dt + \int_0^t \sigma_t d\mathbf{W}_t, \quad (2)$$

$$\mathbf{X}_t^D = \int_0^t \int_{|x| \leq 1} \delta(s, x)(\mu - \nu)(ds, dx) + \int_0^t \int_{|x| > 1} x \mu(ds, dx), \quad (3)$$

⁴ See, e.g. Andersen et al., 2007; Mancini, 2009; Lee and Mykland, 2008 among many for related discussions.

⁵ Same as in Jacod and Todorov (2009), we define co-jumps as two jumps occur at the same time. However, we define idiosyncratic or individual jumps as two jumps occur not at the same time. So that in the current study common jumps plus idiosyncratic jumps equals total jump.

⁶ For example, Lee (2012) is one of the latest works in the literature discussing the possibility of disentangling as well as predicting these two types of jumps in the stock markets. The work holds true that distinction between systematic jumps and idiosyncratic jumps is expected to be beneficial in portfolio management as it can help to better understand the determinants of non-diversifiable risk in extreme market conditions.

⁷ It is generally believed that jump occurs more frequently and dramatically in emerging markets and we use Chinese data just for illustration purpose. Applying the proposed estimator to other markets is not exclusive.

⁸ See, e.g., Delbaen and Schachermayer, 1994.

where $b = (b_t)_{t \geq 0}$ and $\delta = (\delta_t)_{t \geq 0}$ are d -dimensional predictable locally bounded processes, $\sigma = (\sigma_t)_{t \geq 0}$ is an adapted càdlàg $d \times d$ co-volatility matrix and $W = (W_t)_{t \geq 0}$ is d -dimensional Brownian motion, μ is two-dimensional jump measure compensated by ν , and the component of ν has the form $dtF_{it}(dx)$, for $i = 1, 2, \dots, d$, where $F_{it}(dx)$ is a transition measure form $\Omega \times \mathbb{R}_+$ endowed with the predictable σ -field into $\mathbb{R}/0$. The integrated covariation, which is of interest, is then defined as

$$[X_i^c, X_j^c]_T = \int_0^T \sigma_{is} \sigma_{js} ds. \quad (4)$$

For this multi-dimensional process, a quantity which is called the quadratic covariation matrix, is pivotal in financial economics.⁹

2.2. The effect of microstructure noise

We consider the one-dimensional process Y on the time interval $[0, t]$, which can be observed in the following form:

$$Y_{t_i} = X_{t_i} + \epsilon_{t_i}, \quad (5)$$

where X_{t_i} and ϵ_{t_i} , $i = 1, 2, \dots, n$, have straightforward interpretations in terms of efficient price and microstructure noise contamination in the price data, respectively. Both the efficient (latent) price X and microstructure noise ϵ are unobservable. The econometrician only observes the noisy price data Y .

The more sophisticated construction of the noise process follows the work of Christensen et al. (2010), here we need an assumption on the two-dimensional noise process ϵ for technical reason.

Assumption 1. $\epsilon_1(t)$ and $\epsilon_2(t)$ are two iid processes with $E[\epsilon_i(t)] = 0$, $E[\epsilon_1(t)\epsilon_2(t)] = 0$, and $\begin{cases} E[\epsilon_1(t)] = \omega_1^2 \\ E[\epsilon_2(t)] = \omega_2^2 \end{cases}$. We further assume $X \perp \epsilon$ (here, \perp denotes stochastic independence).

The usually used method to remove the microstructure noise effect, is the pre-averaging approach. For a detailed description, see Jacod et al. (2009). Let the j th increment of a process Y is

$$\Delta_j^n Y := Y_{t_j} - Y_{t_{j-1}}, \quad \text{for } j = 1, 2, \dots, n.$$

Hence we have a sequence $(\Delta_1^n Y, \dots, \Delta_n^n Y)$. Choose an integer k_n such that $1 \leq k_n \leq n$ and then we formulate $n - k_n + 1$ overlapping blocks, the i th being

$$B_i = (\Delta_i^n Y, \dots, \Delta_{i+k_n-1}^n Y), \quad \text{for } 1 \leq i \leq n - k_n + 1.$$

Within this i th block, we take a weighted average of the increments:

$$\Delta_{i,k_n}^n Y(g) = \sum_{j=1}^{k_n-1} g(j/k_n) \Delta_{i+j}^n Y, \quad \text{for } 1 \leq i \leq n - k_n + 1,$$

where the weight function g is chosen such that

- (1) it is continuous, piecewise C^1 with a piecewise Lipschitz derivative g' ,
- (2) $g(s) = 0$ when $s \notin (0, 1)$, and $\int_0^1 g^2(s) ds > 0$.

The following two functions satisfy the above conditions

$$g_1(x) = \min\{x, 1 - x\}, \quad \text{and} \quad g_2(x) = x(1 - x), \quad x \in [0, 1].$$

2.3. Threshold quadratic variation with microstructure noise

The above pre-averaging procedure reduces the influence of the microstructure noise; its effect on the jumps remains very limited. After pre-averaging, the estimator contains a part from jumps. The remaining task is to remove the effect of the jumps.

For ease of exposition, let us take $k_n = \theta n^{1/2}$ with a real number θ (the optimal choice of θ is discussed in Christensen et al. (2010)). As we can see in Appendix A.1, after the pre-averaging procedure, we see that smoothed increments, A_1^i and A_2^i , respectively from the diffusion part and the smooth noise part, are both of size $n^{-1/4}$. However, the smoothed increment, A_2^i , from the jump part, may still be larger than $n^{-1/4}$. Following the hypothesis of Mancini (2009) or Jacod (2008), we can propose a noise version of threshold estimator of $[X^c, X^c]_t$ as follows:

$$U(Y, g)_t^n = \sum_{i=1}^{n-k_n+1} (\Delta_{i,k_n}^n Y(g))^2 \mathbf{1}_{\{|\Delta_{i,k_n}^n Y(g)| \leq u_n\}}, \quad (6)$$

where u_n satisfying

$$u_n/n^{-\varpi_1} \rightarrow 0, \quad u_n/n^{-\varpi_2} \rightarrow \infty, \quad \text{for some } 0 \leq \varpi_1 < \varpi_2 < \frac{1}{4}. \quad (7)$$

The theoretical properties of above estimator are studied by Jing et al. (2010). Note that the threshold level u_n is chosen such that those (smoothed) increments larger than u_n will be gradually excluded as $n \rightarrow \infty$, and essentially only those increments due to continuous part and microstructure noise are included. Since the microstructure noise is easy to pick out using realized volatility, hence the integrated volatility is obtained.

2.4. Threshold-pre-averaging estimator of co-volatility

We now consider the estimate of integrated co-volatility, which allows two jump-processes to be correlated. The two individual stocks may jump together because common news such as government announcements may affect individual stocks. We refer to Jacod and Todorov (2009).

After the pre-averaging procedure, the smoothed increments from the diffusion part and the smoothed noise part, are both of size $\Delta_n^{1/4}$, while the smoothed increments from the jump part may still be larger than $\Delta_n^{1/4}$. Suppose that we observe two process $Y = (Y_1, Y_2)$ at time points t_i , $i = 1, 2, \dots, n$. Following the idea of Mancini (2009) or Jacod (2008), we can construct a consistent estimator using two thresholds:

$$U(Y) = \sum_{i=0}^{n-k_n+1} (\Delta_{i,k_n}^n Y_1) (\Delta_{i,k_n}^n Y_2) \mathbf{1}_{\{|\Delta_{i,k_n}^n Y_1| \leq u_n^{(1)}\}} \mathbf{1}_{\{|\Delta_{i,k_n}^n Y_2| \leq u_n^{(2)}\}}, \quad (8)$$

where $u_n^{(k)}$ satisfies

$$u_n^{(k)}/\Delta_n^{\varpi_1} \rightarrow 0, \quad u_n^{(k)}/\Delta_n^{\varpi_2} \rightarrow \infty, \quad \text{for some } 0 \leq \varpi_1 < \varpi_2 < 1/4, \quad k = 1, 2. \quad (9)$$

The threshold level $u_n^{(k)}$ is chosen such that those (smoothed) increments larger than $u_n^{(k)}$ will be gradually excluded as $n \rightarrow \infty$, and essentially only those increments due to continuous part (and microstructure noise part when estimating the integrated volatility) are included for the calculation of the integrated co-volatility because we have already smoothed the data by pre-averaging.

Combining these two methods, we have the next theorem similar to Jing et al. (forthcoming), which is needed here to induce our estimator for non-synchronous observations.

Theorem 1. Assume that $E\|\epsilon\|^4 < \infty$. Under Assumption 1, we have

$$\frac{1}{\theta g(2)} \Delta_n^{1/2} U(Y) \xrightarrow{P} \int_0^T \sigma_1(s) \sigma_2(s) ds. \quad (10)$$

⁹ See, e.g., Barndorff-Nielsen and Shephard, 2007; Andersen et al., 2010 and it is of strong interest in financial applications, such as portfolio risk and hedging of funds management, see, e.g., Ait-Sahalia and Jacod, 2010

The asymptotic properties of the estimator are provided in Appendix A.

2.5. Estimate of co-volatility with non-synchronous data

Non-synchronous trading is a recognized feature in the market. This is especially the case for multivariate high-frequency data. For example, see Fisher (1966). Appearance of this feature causes spurious cross-autocorrelation amongst asset price returns sampled at regular intervals in calendar time, as new information gets built into prices at varying intensities. On the other hand, it is well known that high-frequency realized covariance estimates, using the previous-tick method to align prices, are biased towards zeros in this setting, which is known as Epps effect, see Epps (1979). Motivated by these shortcomings of realized covariance, a number of alternative procedures have been proposed in the literature; to name a few, see Zhang (2011) and Barndorff-Nielsen et al. (2008).

Among others, it is worth pointing out that Hayashi and Yoshida (2005) proposed (and further developed the method in Hayashi and Yoshida (2008)) an alternative procedure for covariance measurement in a noise-free situation. The estimator they propose has a profound advantage in that it does not discard information that is typically lost using a synchronization procedure. Christensen et al. (2010) extend their estimator to noisy high-frequency data using pre-averaging. Here, we show how one can remove the jumps from HY estimator by a threshold procedure.

The combination of those three techniques, namely, threshold, pre-averaging and HY procedures, produces a consistent estimator of co-volatility. Suppose that the number of observations of Y_1 and Y_2 are n_1 and n_2 , respectively. Letting $n = n_1 + n_2$, we define

$$JHY[Y] = \frac{1}{(k_n \int_0^1 g(x) dx)^2} \sum_{i=0}^{n_1-k_n+1} \sum_{j=0}^{n_2-k_n+1} (\Delta_{i,k_n}^n Y_1) (\Delta_{j,k_n}^n Y_2) \times \mathbf{1}_{\{|\Delta_{i,k_n}^n Y_1| \leq u_n^{(1)}, |\Delta_{j,k_n}^n Y_2| \leq u_n^{(2)}, A_{ij} \neq \emptyset\}}, \quad (11)$$

where $A_{ij} = (t_i^{(1)}, t_{i+k_n}^{(1)}) \cap (t_j^{(2)}, t_{j+k_n}^{(2)})$. The indicator function discards pre-averaging returns that do not overlap in time and that the sizes are larger than the thresholds.

Theorem 2. Assume that $E\|\epsilon\|^4 < \infty$. Under Assumption 1, we have

$$JHY[Y] \xrightarrow{P} \int_0^T \sigma_1(s) \sigma_2(s) ds. \quad (12)$$

Remark 1. In practice, there might be interactions between the two securities. Considering this fact in the model, they are cancellations between the increments of two processes. It becomes increasingly complex if one takes account of the signs of jumps, hence in this first step, we only examine the continuous parts. If two stocks share a common jump in their increments, that is $\Delta_{i,k_n}^n Y_l(g) = \Delta_{i,k_n}^n Y_l^c(g) + J$ for $l = 1, 2$, when this jump is small with a matched size as continuous parts, then the cancellation between two continuous parts will induce $\Delta_{i,k_n}^n Y_l(g) < u_n^{(l)}$. Hence a more reasonable estimator in the practice is an adjustment of Eq. (11) as follows:

$$\frac{1}{(k_n \int_0^1 g(x) dx)^2} \sum_{i=0}^{n_1-k_n+1} \sum_{j=0}^{n_2-k_n+1} (\Delta_{i,k_n}^n Y_1) (\Delta_{j,k_n}^n Y_2) \times \mathbf{1}_{\{|\Delta_{i,k_n}^n Y_1| \leq u_n^{(1)}, |\Delta_{j,k_n}^n Y_2| \leq u_n^{(2)}, |\Delta_{i,k_n}^n Y_1| + |\Delta_{j,k_n}^n Y_2| \leq u_n^{(c)}, A_{ij} \neq \emptyset\}}, \quad (13)$$

where $u_n^{(l)}$ is same as in Eq. (11) ($l = 1, 2$), $u_n^{(c)}$ is a positive number but less than $u_n^{(1)} + u_n^{(2)}$, and has the same order as $u_n^{(1)} + u_n^{(2)}$. How-

ever, it is worth pointing out that there is no difference theoretically for the two estimators.

3. Simulation study

In this section, we conduct a simulation study to verify the performance of our estimator with simulated non-synchronous high-frequency trading data. Two processes are generated from a combination of diffusion and symmetric stable Levy process, namely,

$$X_{1t} = \sigma_1 W_{1t} + S_{1t}, \quad \text{and} \quad X_{2t} = \sigma_2 W_{2t} + S_{2t},$$

where W_{1t} and W_{2t} are two standard Brownian motions with correlation ρ , and S_{1t} and S_{2t} are two independent symmetric stable Levy processes with index $\gamma_1 = 0.5$ and $\gamma_2 = 0.5$ to avoid too many small jumps, respectively. To generate the trading time, we employ Poisson processes. Suppose that the trading data is recorded in seconds within a 6.5 h trading day. That is, the time between two ticks of trading is assumed to have a geometric random variable with intensities λ_1 and λ_2 , respectively.

Furthermore, we assume that both of the latent prices are contaminated by a sequence of microstructure noise, ϵ_i , for $i = 1, 2$, which are mutually independent and identically distributed as $N(0, \omega)$, and independent of the latent prices. That is, we have observations $Y = X + \epsilon$. To calculate the pre-averaging return, we use the weight function $g(x) = \min\{x, 1 - x\}$ and choose $k_n = \theta \sqrt{n_1 + n_2}$, where n_1 and n_2 are the number of observations for X_1 and X_2 , respectively and θ is a known constant. The threshold to remove the jumps is $u_n^{(i)} = \alpha_i (1/n_i)^{\kappa_i}$, $i = 1, 2$. In this simulation, we take $\alpha_1 = \alpha_2 = 5$, $\omega = 0.05$, $\theta = 0.8$ and $\kappa_1 = \kappa_2 = 0.23$, more judg-

matistical choice of κ_i may depend on the index of Levy process γ and we consider several cases, where different λ 's are involved.

In particular, to determine the contribution of the jumps and microstructure noise clearly, we use the following estimators to the noisy data:

I. **Realized co-volatility:** An estimator without any treatment of jumps and microstructure noise;

$$RV = \sum_{i,j} \Delta_i Y_1 \Delta_j Y_2 \mathbf{1}_{\{[t_i, t_{i+1}] \cap [t_j, t_{j+1}] \neq \emptyset\}}.$$

II. **Thresholding realized co-volatility:** An estimator considering the effect of jumps only;

$$TRV = \sum_{i,j} \Delta_i Y_1 \Delta_j Y_2 \mathbf{1}_{\{|\Delta_i^n Y_1| \leq \Delta_1^{1/2-\epsilon_1}, |\Delta_j^n Y_2| \leq \Delta_2^{1/2-\epsilon_2}, [t_i, t_{i+1}] \cap [t_j, t_{j+1}] \neq \emptyset\}},$$

where $\Delta_1 = \frac{1}{n_1}$ and $\Delta_2 = \frac{1}{n_2}$.

III. **Pre-averaging realized co-volatility:** An estimator considering the effect of microstructure noise only, i.e., Christensen et al. (2010):

$$HY[Y] = \frac{1}{(k_n \int_0^1 g(x) dx)^2} \sum_{i=0}^{n_1-k_n+1} \sum_{j=0}^{n_2-k_n+1} (\Delta_{i,k_n}^n Y_1) (\Delta_{j,k_n}^n Y_2) \mathbf{1}_{\{A_{ij} \neq \emptyset\}},$$

where $A_{ij} = (t_i^{(1)}, t_{i+k_n}^{(1)}) \cap (t_j^{(2)}, t_{j+k_n}^{(2)})$.

IV. **Thresholding-pre-averaging realized co-volatility:** An estimator considering both effects, i.e., the estimator we proposed:

$$THY[Y]_{rl} = \frac{1}{(k_n \int_0^1 g(x) dx)^2} \sum_{i=0}^{n_1-k_n+1} \sum_{j=0}^{n_2-k_n+1} (\Delta_{i,k_n}^n Y_1) \times (\Delta_{j,k_n}^n Y_2) \mathbf{1}_{\{i \in B_i^n, j \in C_j^n, A_{ij} \neq \emptyset\}},$$

where $B_i^n = \{i : |\Delta_{i,k_n}^n Y_1| \leq u_n^{(1)}\}$, $C_j^n = \{j : |\Delta_{j,k_n}^n Y_2| \leq u_n^{(2)}\}$.

Table 1

Contribution of jumps, microstructure noise to the co-volatility.

	$\rho = 0.5$ (bias, rmse)	$\rho = 0$ (bias, rmse)	$\rho = -0.5$ (bias, rmse)
Panel A: "Estimator" I: Neither			
$\lambda_1 = 1, \lambda_2 = 2$	(-12.452, 10.221)	(-6.3254, 8.6698)	(-8.3698, 3.5875)
$\lambda_1 = 3, \lambda_2 = 5$	(-23.458, 15.874)	(-24.365, 19.365)	(-14.2157, 7.5687)
$\lambda_1 = 4, \lambda_2 = 6$	(5.2654, 6.1258)	(17.875, 9.3636)	(51.245, 15.332)
$\lambda_1 = 8, \lambda_2 = 10$	(8.3365, 9.3654)	(26.258, 18.856)	(23.654, 13.558)
$\lambda_1 = 12, \lambda_2 = 20$	(-42.078, 30.254)	(-30.487, 26.354)	(-45.265, 16.325)
Panel B: "Estimator" II: Jump			
$\lambda_1 = 1, \lambda_2 = 2$	(0.2154, 0.1678)	(0.2254, 0.2142)	(0.2241, 0.3245)
$\lambda_1 = 3, \lambda_2 = 5$	(0.3278, 0.3587)	(0.3354, 0.4157)	(0.3125, 0.5236)
$\lambda_1 = 4, \lambda_2 = 6$	(0.5547, 0.5786)	(0.5811, 0.7698)	(0.4299, 0.7865)
$\lambda_1 = 8, \lambda_2 = 10$	(0.8975, 0.6687)	(0.9856, 0.7752)	(0.6411, 0.8967)
$\lambda_1 = 12, \lambda_2 = 20$	(1.8956, 0.8679)	(1.2356, 1.0015)	(0.9654, 1.4658)
Panel C: "Estimator" III: Noise			
$\lambda_1 = 1, \lambda_2 = 2$	(-0.1986, 0.8658)	(0.2542, 0.6636)	(-0.2687, 0.9865)
$\lambda_1 = 3, \lambda_2 = 5$	(-0.5425, 1.3254)	(-0.3152, 2.1354)	(0.5645, 2.3457)
$\lambda_1 = 4, \lambda_2 = 6$	(0.6225, 1.8869)	(0.7541, 2.0124)	(-0.4299, 1.6752)
$\lambda_1 = 8, \lambda_2 = 10$	(0.5521, 2.2021)	(-0.5096, 1.6589)	(0.9857, 2.1925)
$\lambda_1 = 12, \lambda_2 = 20$	(-0.7711, 0.8568)	(0.6568, 2.9854)	(-1.0554, 2.2125)
Panel D: "Estimator" IV: Both			
$\lambda_1 = 1, \lambda_2 = 2$	(-0.0148, 0.1314)	(-0.0102, 0.1142)	(-0.0124, 0.1441)
$\lambda_1 = 3, \lambda_2 = 5$	(-0.0243, 0.1521)	(-0.0211, 0.1354)	(-0.0245, 0.1588)
$\lambda_1 = 4, \lambda_2 = 6$	(-0.0398, 0.1713)	(-0.0412, 0.1524)	(-0.0299, 0.1752)
$\lambda_1 = 8, \lambda_2 = 10$	(-0.0521, 0.2021)	(-0.0496, 0.1752)	(-0.0411, 0.1968)
$\lambda_1 = 12, \lambda_2 = 20$	(-0.0711, 0.2867)	(-0.0568, 0.2144)	(-0.0654, 0.2241)

Note: This table reports the results from our simulation example. "Neither" refers to when neither noise nor jump is controlled for; "Jump" and "Noise" refer to estimators when only jump or noise is considered and controlled for respectively; "Both" refers to when both noise and jump are controlled. The corresponding contribution of jumps, microstructure noise can be clearly obtained from the table.

All of the results are displayed for various choices of parameter λ in Table 1. The same procedure is repeated 5000 times and results including biases and mean square errors are displayed in several tables.

From the table, we can illustrate the following findings: the threshold-pre-averaging estimators perform very well across all scenarios of noise and non-synchronous trading. One exception is the threshold-pre-averaging estimator in which all of the other "estimators" actually wrongly estimate the co-volatility; however, this is in line with our theoretical results. That is, we have to consider both the jumps and microstructure noise effects. Otherwise, a bias might be induced. From the "estimator" I, we find that the microstructure noise, if left untreated, will play a dominant role in the statistic.

4. Empirical results

In our empirical illustration, we choose the Chinese HUSHEN300 index and four blue chip Chinese stocks. Of the four stocks, three are from the Banking sector and one is from Iron and Steel sector. In symbols, HS300 stands for HUSHEN 300 index; SPDB stands for Shanghai PuDong Development Bank; HXB stands for Huaxia Bank; CMSB stands for China Minsheng Bank; and WG

stands for Wuhan Iron and Steel (Group) Corp. We obtain tick-by-tick high frequency data for the four stocks and the HUSHEN 300 index from the China Stock Market and Accounting Research Database.

Similar to the filter rules used by Barndorff-Nielsen et al. (2008) and Christensen et al. (2010), we filter the data by choosing those from 9:30 AM to 3:00 PM. We also aggregate data with identical time stamps using volume-weighted average prices. Table 2 shows the numbers of observations for the raw trades and filtered trades in 2009 and 2010.

Following the settings in the simulation study, we compute the sample co-volatility matrix for the five assets (HUSHEN 300 index and the four stocks) in 2009 and 2010 respectively under the four scenarios of (I)–(IV), corresponding to "Neither", "Jump", "Noise" and "Both" respectively.

The two thresholds $u_n^{(1)}$ and $u_n^{(2)}$ are used in the estimators of (II) and (IV). To choose them, we assume that the pre-averaging returns are normally distributed if no jumps are included. The use of tick data makes the sample size large enough. If there are jumps in the pre-averaging returns, they must be outliers. As a consequence, we select $u_n^{(1)}$ and $u_n^{(2)}$ as seven times standard deviations of each data set (because a normal variable is almost impossible with size larger than 7 st.dev.). Additionally, in such empirical investigation, we rely on the third threshold of u_n^c for disentangling the common jumps from the idiosyncratic jumps and to treat exceeding of this threshold as an occurrence of common jumps between the two assets.¹⁰ Accounting for the cancelations discussed in the estimator of Eq. (13), we reduce one standard deviation to the lower bound of the common jumps. That is, we set $u_n^{(c)}$ equal to $6/7 \times (u_n^{(1)} + u_n^{(2)})$.

It is clear that scenario (III) is simply the estimator proposed by Christensen et al. (2010) while scenario (IV) is the estimator proposed by the current study. The 2 year results, which are averaged co-volatility estimated over each year, are presented in Tables 3 and 4 respectively. The diagonal is the averaged variance of the five with the rest being calculated integrated covariances. From the results for year 2009 in Table 3, if we compare the co-volatilities across panels, compared to controlling for neither noise nor jumps, we can easily obtain the following observations: first, controlling for jumps only yields a smaller realized variance estimation and a comparatively even smaller realized covariance estimation; second, controlling for noise only yields smaller integrated realized variances but much larger realized covariances¹¹; and third, accounting for noise and jumps simultaneously will produce smaller realized variance estimation and larger realized covariance estimation, but not as large as from the observation two above.

From these observations, one might easily understand that jumps, if not controlled for, usually lead to increased estimation for the realized covariances while microstructure noises usually lead to decreased estimations. Controlling different aspects will, however, produces smaller integrated realized variances, which is not surprising given our thresholding techniques applied in the present study. Additionally, comparing matrix based on our estimator to that of Christensen et al. (2010), i.e. (IV)–(III), we notice that our estimator generally produce slightly smaller realized variances but much smaller realized covariances.

Based on our analysis, almost the same relative cross-panel patterns for the co-volatility matrices are observed from Table 4 for

¹⁰ Such a threshold can realistically account for those co-jumping scenarios when both assets exceed the two thresholds together and when both assets jump but none exceeds the threshold as a result of interactions between different components of assets.

¹¹ This is consistent with the result in Table 7 for realized beta discussion that controlling for noise only will render beta going up significantly. And patterns for all the four aspects here for co-volatility can also adequately explain patterns observed for the estimated realized betas.

Table 2
Raw trades and filtered trades.

	2009					2010				
	HS300	SPDB	WG	HXB	CMSB	HS300	SPDB	WG	HXB	CMSB
Raw trades	796,999	800,596	769,289	803,029	795,158	688,441	688,622	686,965	679,565	691,681
Filtered trades	673,317	672,683	669,977	675,450	667,240	670,404	666,083	665,372	659,016	668,888

Note: this table reports the raw and filtered data for the selection of trades included in our empirical sample in years 2009 and 2010.

Table 3
Average of co-volatility matrices across the trading days in 2009: Under four scenarios.

	Panel A: Neither					Panel B: Jump				
	HS300	SPDB	WG	HXB	CMSB	HS300	SPDB	WG	HXB	CMSB
HS300	0.4994	0.5147	0.5014	0.4211	0.5170	0.3904	0.2370	0.2386	0.1999	0.2742
SPDB	–	13.1669	0.6130	0.5422	0.5090	–	6.8450	0.2672	0.2184	0.2215
WG	–	–	29.9545	0.5831	0.5645	–	–	20.3575	0.2830	0.2402
HXB	–	–	–	21.9932	0.5453	–	–	–	13.0184	0.3005
CMSB	–	–	–	–	32.7977	–	–	–	–	25.8879
Panel C: Noise						Panel D: Both				
HS300	1.7917	3.7717	3.8462	3.3597	3.3651	1.5335	2.4765	2.5355	2.1323	2.2308
SPDB	–	7.6661	4.2855	4.5801	4.4861	–	6.5996	2.6739	2.6442	2.5927
WG	–	–	10.1980	3.6886	3.7442	–	–	8.6004	2.2519	2.3938
HXB	–	–	–	4.0540	4.1606	–	–	–	6.2495	2.3474
CMSB	–	–	–	–	3.1546	–	–	–	–	3.1279

Note: This table presents the co-volatility for the market index and the four stocks in 2009 in four scenarios to compare the contributions of different aspects. “Neither” refers to when neither the noise nor the jumps are controlled for in the estimation; “Jump” refers to when only jumps are controlled for; “Noise” refers to when only microstructure noise is controlled for; “Both” refers to when both noise and jumps are accounted for although all four aspects have already considered the non-synchronous trading problem. Clearly, Panel C gives the estimate based on Christensen et al. (2010) while Panel D is based on the new estimator we proposed. The results are in the units of 10^{-4} .

Table 4
Average of co-volatility matrices across the trading days in 2010: Under four scenarios.

	Panel A: Neither					Panel B: Jump				
	HS300	SPDB	WG	HXB	CMSB	HS300	SPDB	WG	HXB	CMSB
HS300	0.3908	0.3387	0.2870	0.3503	0.3790	0.2687	0.1441	0.1720	0.1498	0.2161
SPDB	–	12.0413	0.3099	0.4115	0.3334	–	7.1831	0.1707	0.1610	0.1412
WG	–	–	39.0687	0.3759	0.2619	–	–	35.6797	0.1925	0.1529
HXB	–	–	–	19.8757	0.3487	–	–	–	10.6351	0.1772
CMSB	–	–	–	–	33.8252	–	–	–	–	29.3817
Panel C: Noise						Panel D: Both				
HS300	1.1014	2.1490	1.9238	2.0714	1.6321	0.8462	1.2177	1.1461	1.1712	0.9301
SPDB	–	4.0676	2.0834	2.7062	2.0361	–	3.1898	1.1374	1.2896	0.9937
WG	–	–	3.0930	1.9645	1.6227	–	–	2.4869	1.0893	0.8901
HXB	–	–	–	3.7894	2.0211	–	–	–	2.7342	0.9675
CMSB	–	–	–	–	1.7490	–	–	–	–	1.1973

Note: This table presents the co-volatility for the market index and the four stocks in 2010 in four scenarios to compare the contributions of different aspects. “Neither” refers to when neither the noise nor the jumps are controlled for in the estimation; “Jump” refers to when only jumps are controlled for; “Noise” refers to when only microstructure noise is controlled for; “Both” refers to when both noise and jumps are accounted for although all four aspects have already considered the non-synchronous trading problem. Clearly, Panel C gives the estimate based on Christensen et al. (2010) while Panel D is based on the new estimator we proposed. The results are in the units of 10^{-4} .

the 2010 as from Table 3. But compare the two tables, the matrix value in 2010 is generally smaller than that in 2009, indicating the estimated co-volatilities are weaker in magnitude than in the previous year. Taking the last panel when both noise and jumps effects are controlled for example, for the market index, variance estimation in 2010 is almost halved comparing to the previous year. In 2009, WG, the Steel and Iron company, shows the biggest variation in terms of realized variance at 8.6004. This figure is approximately twice that of the market and about 2.5 times of that of the CMSB (3.1279), which is the smallest among all four. But in 2010, WG’s volatility is obviously reduced to be lower than even one of the big banks (HXB, 2.7342), while CMSB is still the lowest in terms of variance. This reflects that the majority of the banking stocks, due to their size and regulation, shows relatively smaller variance while industrial stocks in China, such as WG, will fluctuate more subject to various market and trading influences. How-

Table 5
Contribution of jumps to volatility (thresholding based) (%).

	HS300	SPDB	WG	HXB	CMSB
2009	14.10	13.54	16.03	15.69	15.51
2010	22.89	19.59	16.50	18.68	18.78

Note: This table presents analysis results of jumps’ contribution to realized variance based on our thresholding methodology. The ratios are averaged across trading days in 2009 and 2010. Jumps roughly contribute about 15% to total realized variances in 2009 with almost equal proportions for index and all the four individual stocks. In 2010, such ratio is about 19%, which suggests total jumps contribute a little bit more in 2010 than in 2009.

ever, covariances estimated is similar in magnitude across pairs within each year, confirming that linkages between them are fundamentally and commonly determined.

Table 6

Breakdown of the jump effect on the integrated co-volatility estimation (%).

	Panel A: 2009				Panel B: 2010			
	SPDB	WG	HXB	CMSB	SPDB	WG	HXB	CMSB
Continuous	83.61	84.96	82.65	79.49	74.70	63.04	73.28	59.57
Common jumps	15.25	13.24	15.11	13.55	22.72	14.38	22.55	16.89
Idiosyncratic jumps	1.14	1.80	2.24	6.96	2.58	22.58	4.17	23.54

Note: This table presents the breakdown of jumps effect on the co-volatility with the HUSHEN300 index for the four stocks in 2009 and 2010. The quotients are based on Christensen et al. (2010) estimate as 100%, which equals to continuous part plus total jump part, which is further decomposed into common jumps and idiosyncratic jumps. The percentage reported in the table is proportion of the indicated component to the Christensen et al. (2010) estimate. The three components add up to be one.

Table 7

The average value of the four betas for 2009 and 2010.

	Estimated beta				Effects of aspects (%)			
	SPDB	WG	HXB	CMSB	SPDB	WG	HXB	CMSB
Panel A: 2009								
Neither	0.9705	0.9512	0.7976	0.9832	44.47	42.30	41.03	49.80
Jump	0.6467	0.6681	0.5465	0.7819	29.63	29.71	28.12	39.60
Noise	2.1823	2.2489	1.9438	1.9744	–	–	–	–
Both	1.6928	1.7401	1.4742	1.5122	77.57	77.38	75.84	76.59
Panel B: 2010								
Neither	0.8339	0.7294	0.8647	0.9928	39.62	38.27	42.32	59.66
Jump	0.5187	0.6336	0.5284	0.8436	24.64	33.25	25.86	50.70
Noise	2.1047	1.9057	2.0431	1.6640	–	–	–	–
Both	1.5442	1.4797	1.4773	1.2411	73.37	77.65	72.31	74.59

Note: This table presents the realized beta for the four stocks in the 2 years from four aspects to compare the contributions. “Neither” refers to when neither the noise nor the jumps are controlled for in the estimation, although all four aspects have already considered the non-synchronous trading problem. Clearly, aspect “Noise” gives the estimate based on Christensen et al. (2010) while “Both” is based on the new estimator we proposed. Those effects of aspects are proportions which is calculated as ratios for different aspects against (III), i.e. Christensen et al. (2010) estimator.

We also investigate the effect of jumps on volatility and co-volatility within the newly proposed methodological framework to try to better understand how different jump activities contribute to the estimation of the co-volatility. Table 5 presents analysis results of the contribution of jumps to realized variance based on our thresholding methodology. We compute the realized variances under Christensen et al. (2010) estimator and our own estimator, and take the difference between the two as component of total jump. The contribution proportions are computed as the ratio between absolute value of the jumps to realized variance based on the Christensen et al. (2010) estimator. The ratios are then averaged across trading days in 2009 and 2010. From Table 5, jumps roughly contribute about 15% of the total realized variances in 2009 with comparable proportions for the index and the four individual stocks. In 2010, such ratio is near 19%, which suggests a total jumps contributing slightly more in 2010 than in 2009.¹²

We directly report the three components of the co-volatility estimates as proportions of the Christensen et al. (2010) estimator when noise and non-synchronous trading are already controlled for. The three components are thus the continuous part, common jumps, and idiosyncratic jumps, in which the last two components add up to be the total jumps. Following Jacod and Todorov (2009), we define common jumps as two assets jump together and the rest of jumping activities are simply idiosyncratic jumps.¹³

From Table 6, we present the breakdown of jump effect on the integrated co-volatility estimation for the stocks with the HUSH-

EN300 index. Panel A indicates that in 2009, continuous part made up approximately 82% of the integrated co-volatility. This suggests total jumps take about 18% of the estimate, among which common jumps clearly dominate to show approximately 15% while idiosyncratic jumps takes the remaining 3%. The proportion of the idiosyncratic jumps is the highest for the CMSB at 7% and lowest for SPDB at 1.1%. All the results generally suggest that common jumps dominate idiosyncratic jumps in the total jump effect in the integrated co-volatility estimation in 2009. This should be straightforward as all the four stocks are blue chips and one can expect them to jump mostly together with the index. Panel B indicates that continuous parts have a relatively lower proportion in 2010 than in 2009, although they still average at about 68%. Regarding the common jump and idiosyncratic jump proportions, SPDB and HXB continue to show that common jumps dominate the total jump effect while WG and CMSB results suggest idiosyncratic jumps dominate, although corresponding common jumps still occupy a nontrivial proportion. The proportions of the continuous parts for when the idiosyncratic jumps dominate are relatively lower although. In total, results of the jump effect decomposition from Table 6 generally indicate that common jump dominates the total jump effect on the co-volatility estimation. However, it is also possible for idiosyncratic jumps to take control, depending on the jumping activities of individual stocks. Results from our study here also suggest it is important to consider both common jumps and idiosyncratic jumps in the risk return analysis as both can affect risk measurement significantly.

We present average beta estimated across trading days in 2009 and 2010 under the four scenarios for the four stocks, together with the effect of each scenario, in Table 7. From both panels, by considering the four scenarios we obtain that: firstly, when neither noise nor jump is controlled for, realized betas estimated for the four stocks are close to one, indicating those stocks appear to move in lockstep with the market index and this is the case in both years. Secondly, controlling for jumps alone, realized beta estimated decrease to around 0.6. This may imply that noise affects the beta estimation leading to possibly downward biases. Thirdly, controlling for microstructure noises, realized beta will be significantly increased to around two for the four stocks. This also indicates jumps will cause considerable upward bias in the beta estimation as noise effects have been considered and controlled for. Furthermore, this indicates beta estimation based on Christensen et al. (2010) may lead to upward bias and overestimation. In the last scenario when both noise and jump effects are considered and controlled for, realized beta estimated are approximately 1.6 in 2009 and 1.4 in 2010, suggesting noise and jumps, if not controlled for in the systematic risk estimation, may lead to considerable underestimation. Consistent results are obtained for the four stocks in both of the 2 years, further confirming the robustness of the findings. The second table in each panel further presents the effect of each aspect as a proportion to the third beta estimator which is based on Christensen et al. (2010).¹⁴ It can be inferred from the panel that total jumps contrib-

¹² It can also be consistently found from Table 6 that total jumps also affect the co-volatility estimations more in 2010 than in 2009.

¹³ It should be noted that we take absolute values of each of the components in the proportion calculation, ensuring that the sum of the proportions is equal to one.

¹⁴ We choose Christensen et al. (2010) estimator as a benchmark on considerations that we need to maintain consistency with results from Table 6 and also for possibly computing the contribution of jumps to beta estimation.

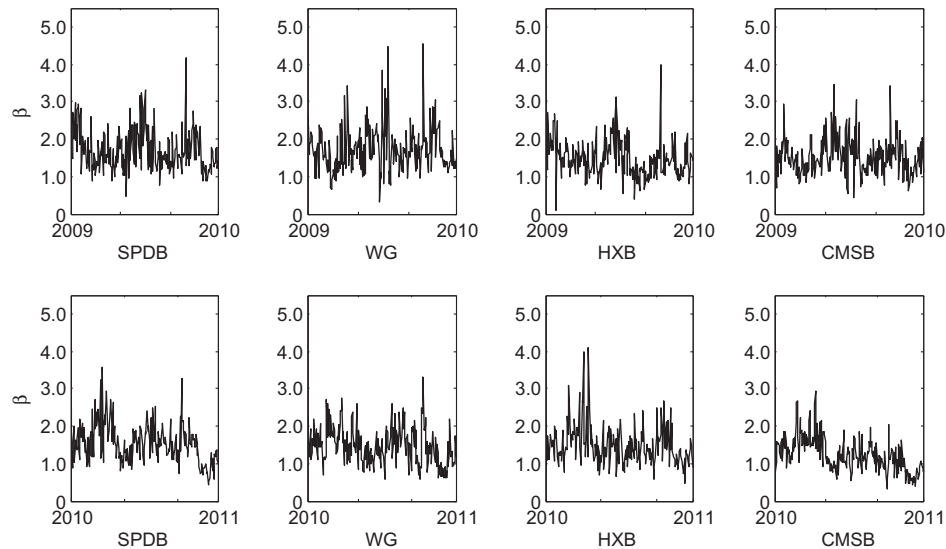


Fig. 1. The estimated betas under scenario (IV). *Note:* This figure plots the estimated betas of the four stocks controlled for the noise and jump. The upper row indicates the year 2009 while the lower row indicates the year 2010. The first column depicts the daily beta of SPDB, the rest columns, from left to right, depicts the daily betas of WG, HXB, CMSB respectively.

ute around 25% to the beta estimation, which is one minus the last row of the proportions. This is comparable with total jumps' contribution to integrated co-volatility estimation documented before. Fig. 1 illustrates the plot of the daily estimated betas of the four stocks under the fourth scenario when both noise and jump effects are considered and controlled for. The upper row depicts the situation for 2009 while the lower row plots the daily beta for 2010. This figure clearly indicates systematic risk is indeed time varying and shows similar cross-sectional patterns within each year and each industry.

The actual realized beta based on our proposed estimator is about 50% higher than otherwise neither noise nor jump is considered and controlled for. Compared with estimator of Christensen et al. (2010), our estimator corrects the overestimation of systematic risk derived from controlling for noise only. The above results are consistent with the rest empirical findings documented in the current study and thus prove the consistency and robustness of our proposed methodology.

5. Concluding remarks

Relying on high frequency data in the estimation for realized beta will have to overcome the problems of microstructure noise and non-synchronous trading, and more importantly the discontinuity of price process, the jumps, which will lead to distortions when estimating integrated co-volatility. Several important studies are proposed dealing with microstructure noise and non-synchronous trading problems such as Christensen et al. (2010), but relatively less attention has been paid to controlling for jumps. This issue is important in that by introducing jumps into the co-volatility estimation, there is the possibility to distinguish between idiosyncratic jump and common jump, and it's important to understand how these different jump activities relate to systematic risk.

We propose a new estimator for integrated co-volatility based on Christensen et al. (2010) but going one step further to control for the effect of jumps. We show how the combination of pre-averaging and thresholding can be applied to the problem of measuring the co-volatility of financial returns under non-synchronous trad-

ing. We investigate the asymptotic properties, such as consistency and asymptotic normality of the proposed realized estimator. We also derive a noise-robust HY estimator that can be implemented on the original data without prior alignment of the prices. The estimator allows very general structure of jumps in the underlying process, for example, infinity activity or even infinity variation. Simulation is included to illustrate the performance of finite sample. After establishing the validity of the estimator, we apply it in disentangling contributions of different aspects on the co-volatility and realized beta estimations using tick-by-tick data from the Chinese stock market. We find that controlling for jumps greatly affect the integrated co-volatility estimation. Common jumps dominate the effect on the estimation of integrated co-volatility and the realized beta. We find it possible for idiosyncratic jumps to affect the co-volatility estimation significantly, which confirm the importance of both jump activities in the estimation. We also find that jumps likely cause upward biases for realized beta estimation while microstructure noise usually lead to downward biases and the combined effects of the two aspects are usually nonlinear and lead to underestimation of systematic risk. By controlling for both noise and jump factors, the proposed new estimator indicates systematic risk estimated should be actually 50% higher than otherwise when neither effect is accounted for. Such results suggest our methodology provides necessary and significant corrections for biases in the co-volatility and realized beta estimation. They also indicate systematic risk for stocks should be much higher than previously estimated which has rich implications in many areas in finance.

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Appendix A

A.1. Review of pre-averaging

Now let us investigate what effect the pre-averaging has. In view of denoting $g_j^n = g(j/k_n)$, we have

$$\begin{aligned} \Delta_{i,k_n}^n Y(g) &= \sum_{j=1}^{k_n-1} g_j^n \Delta_{i+j}^n X^C + \sum_{j=1}^{k_n-1} g_j^n \Delta_{i+j}^n X^D + \sum_{j=1}^{k_n-1} g_j^n \Delta_{i+j}^n \epsilon : \\ &= A_1^i + A_2^i + A_3^i. \end{aligned} \quad (14)$$

Some simple variance calculations show that

$$A_1^i = O_p(\sqrt{k_n/n}), \quad A_3^i = O_p(1/\sqrt{k_n}).$$

(We shall treat the jump component A_2^i in the next subsection.) Clearly, by choosing $k_n \rightarrow \infty$ appropriately, we can control the effect of microstructure noise A_3^i relative to A_1^i . In particular,

- (i) if $k_n/n^{1/2} \rightarrow \infty$, e.g., $k_n = \lfloor cn^{1/2+\epsilon} \rfloor$ for some small $\epsilon > 0$, then the effect of the microstructure noise can be ignored;
- (ii) if $k_n/n^{1/2} = c > 0$, i.e., $k_n = \lfloor cn^{1/2} \rfloor$, then A_1^i and A_3^i will be of comparable size.

In either case, the influence of microstructure noise has been eliminated or substantially reduced.

A.2. Proofs

By a standard localization procedure, see Jacod (2012) for instance, we can replace the local boundedness in assumptions by a boundedness assumption, and also assume that the process Y_i , $i = 1, 2$, and thus the jump process X_{it}^D , are bounded. That is, for all results which need the assumption about volatility and Lévy measure, we may assume, almost surely,

$$\max\{|b_{it}|, |\sigma_{it}|, |X_{it}|\} \leq c, \quad \text{for some constant } c > 0.$$

We more or less use the procedure of Jing et al. (2011).

Proof of Theorem 1. By Theorem 1 of Christensen et al. (2010), we only need to show that

$$\Delta_n^{1/2} (U_{rl}(\mathbf{Y}) - V_{rl}(\mathbf{Y}^C)) \rightarrow^P 0. \quad (15)$$

Rewrite the left hand side as

$$\begin{aligned} \Delta_n^{1/2} (U_{rl}(\mathbf{Y}) - V_{rl}(\mathbf{Y}^C)) &= \Delta_n^{1/2} \sum_{i=0}^{n-k_n+1} \left(\Delta_{i,k_n}^n Y_r^C \right) \left(\Delta_{i,k_n}^n Y_l^C \right) \mathbf{1}_{\{|\Delta_{i,k_n}^n Y_r| \leq u_n^{(r)}, |\Delta_{i,k_n}^n Y_l| \leq u_n^{(l)}\}} - \left(\Delta_{i,k_n}^n Y_r^C \right) \left(\Delta_{i,k_n}^n Y_l^C \right) \\ &+ \Delta_n^{1/2} \sum_{i=0}^{n-k_n+1} \left(\Delta_{i,k_n}^n Y_r^D \right) \left(\Delta_{i,k_n}^n Y_l^D \right) \mathbf{1}_{\{|\Delta_{i,k_n}^n Y_r| \leq u_n^{(r)}, |\Delta_{i,k_n}^n Y_l| \leq u_n^{(l)}\}} \\ &+ \Delta_n^{1/2} \sum_{i=0}^{n-k_n+1} \left(\Delta_{i,k_n}^n Y_r^D \right) \left(\Delta_{i,k_n}^n Y_l^C \right) \mathbf{1}_{\{|\Delta_{i,k_n}^n Y_r| \leq u_n^{(r)}, |\Delta_{i,k_n}^n Y_l| \leq u_n^{(l)}\}} \\ &+ \Delta_n^{1/2} \sum_{i=0}^{n-k_n+1} \left(\Delta_{i,k_n}^n Y_r^C \right) \left(\Delta_{i,k_n}^n Y_l^D \right) \mathbf{1}_{\{|\Delta_{i,k_n}^n Y_r| \leq u_n^{(r)}, |\Delta_{i,k_n}^n Y_l| \leq u_n^{(l)}\}} \\ &=: T_1 + T_2 + T_3 + T_4. \end{aligned}$$

Since the continuous and discontinuous components are independent, it is easy to show that $T_2 \rightarrow^P 0$ and $T_4 \rightarrow^P 0$.

Next, we show $T_3 \rightarrow^P 0$. For any arbitrarily small $\epsilon > 0$, there exists an integer N_1 , as long as $n > N_1$, we have

$$|T_3| \leq \Delta_n^{1/2} \sum_{i=0}^{n-k_n+1} \left| \left(\Delta_{i,k_n}^n Y_r^D \right) \left(\Delta_{i,k_n}^n Y_l^D \right) \right| \mathbf{1}_{\{|\Delta_{i,k_n}^n Y_r| \leq \epsilon/2, |\Delta_{i,k_n}^n Y_l| \leq \epsilon/2\}}.$$

By the Levy law for modulus of continuity of Brownian motion's paths (see, Theorem 9.25, Karatzas and Shreve, 1999) and time changed Brownian motion (Theorems 1.9–1.10, Revuz and Yor, 2001), there exists an integer N_2 , when $n > N_2$, we have

$$\sup_i \frac{\Delta_{i,k_n}^n X_k^C}{\Delta_n^{1/4} \sqrt{2 \log \frac{1}{\Delta_n}}} \leq A(\omega),$$

for $k = r, l$, and some constant A only depending on ω . On the other hand, the central limit theorem implies that

$$\begin{aligned} P\left(|\Delta_{i,k_n}^n \epsilon_k| \leq \sqrt{1/k_n \log \frac{1}{\Delta_n}}\right) &= P\left(|\Delta_{i,k_n}^n \epsilon_k| \leq \Delta_n^{1/4} \log \frac{1}{\Delta_n}\right) \\ &= 1 - o(1/\sqrt{k_n}). \end{aligned}$$

We take $N_3 = \max(N_1, N_2)$ and let $n > N_3$, we get

$$\begin{aligned} |T_3| &\leq \Delta_n^{1/2} \sum_{i=0}^{n-k_n+1} \left| \left(\Delta_{i,k_n}^n Y_r^D \right) \left(\Delta_{i,k_n}^n Y_l^D \right) \right| \mathbf{1}_{\{|\Delta_{i,k_n}^n Y_r| \leq \epsilon, |\Delta_{i,k_n}^n Y_l| \leq \epsilon\}} \\ &+ o_P(1) \rightarrow^P \sum_{s \leq t} |h(g, \theta, s)(\Delta_s Y_r)(\Delta_s Y_l)| \mathbf{1}_{\{|\Delta_s Y_r| \leq \epsilon, |\Delta_s Y_l| \leq \epsilon\}}, \end{aligned}$$

where $h(g, \theta, s)$ is bounded, depends on g, θ and s . Now, Letting $\epsilon \rightarrow 0$ yields $T_3 \rightarrow^P 0$.

Finally, we show that $T_1 =: \Delta_n^{1/2} \sum_{i=0}^{n-k_n+1} T_1^i \rightarrow^P 0$. We consider the following disjoint cases. Note that l_1, l_2, r_1, r_2 denote any positive real numbers.

- If $|\Delta_{i,k_n}^n Y_k^C| \geq u_n^{(k)}/2$, for $k = r, l$, we have for some appropriate constant K ,

$$|T_1^i| \leq \frac{K |\Delta_{i,k_n}^n Y_r^C|^{1+l_1} |\Delta_{i,k_n}^n Y_l^C|^{1+l_2}}{(u_n^{(r)})^{l_1} (u_n^{(l)})^{l_2}}. \quad (16)$$

- If $|\Delta_{i,k_n}^n Y_k^C| \leq u_n^{(k)}/2$, for $k = r, l$, similarly, we have,

$$|T_1^i| \leq \frac{K |\Delta_{i,k_n}^n Y_r^C| |\Delta_{i,k_n}^n Y_l^C| |\Delta_{i,k_n}^n Y_r^D|^{r_1} |\Delta_{i,k_n}^n Y_l^D|^{r_2}}{(u_n^{(r)})^{r_1} (u_n^{(l)})^{r_2}}. \quad (17)$$

- If $|\Delta_{i,k_n}^n Y_r^C| \geq u_n^{(r)}/2$, $|\Delta_{i,k_n}^n Y_l^C| \leq u_n^{(l)}/2$, we have,

$$|T_1^i| \leq \begin{cases} \frac{K |\Delta_{i,k_n}^n Y_r^C|^{1+l_1} |\Delta_{i,k_n}^n Y_l^C| |\Delta_{i,k_n}^n Y_l^D|^{r_2}}{(u_n^{(r)})^{l_1} (u_n^{(l)})^{r_2}}, & \text{if } |\Delta_{i,k_n}^n Y_l| \geq u_n^{(l)}, \\ 0, & \text{if } |\Delta_{i,k_n}^n Y_2| \leq u_n^{(2)} \text{ and } |\Delta_{i,k_n}^n Y_1^C| \leq u_n^{(1)}, \\ \frac{K |\Delta_{i,k_n}^n Y_r^C|^{1+l_1} |\Delta_{i,k_n}^n Y_l^C|}{(u_n^{(r)})^{l_1}}, & \text{if } |\Delta_{i,k_n}^n Y_l| \leq u_n^{(l)} \text{ and } |\Delta_{i,k_n}^n Y_r^C| \geq u_n^{(r)}. \end{cases} \quad (18)$$

- The case of $|\Delta_{i,k_n}^n Y_r^C| \leq u_n^{(r)}/2$, $|\Delta_{i,k_n}^n Y_l^C| \geq u_n^{(l)}/2$ is similar to (18).

Next, we estimate $|\Delta_{i,k_n}^n Y_k^C|$ and $|\Delta_{i,k_n}^n Y_k^D|$. By Holder's and Birkholder's inequalities, we have

$$\begin{aligned} E\left(\left|\Delta_{i,k_n}^n Y_k^D\right|^2\right) &= \int_{i\Delta_n}^{(i+k_n)\Delta_n} \int_R g^2\left(\frac{1}{k_n} \left[\frac{s-i\Delta_n}{\Delta_n}\right]\right) x^2 F_t(dx) ds \leq K(k_n \Delta_n), \\ E\left(\left|\Delta_{i,k_n}^n Y_k^C\right|^l\right) &\leq K_l(k_n \Delta_n)^{l/2} \quad \text{for } l > 0. \end{aligned} \quad (19)$$

Without loss of generality, we let $l_1 = l_2 = r_1 = r_2 = 1$. Then we deduce from above inequalities and estimations that $\Delta_n^{1/2} E(|T_1^i|) \leq K \left(\frac{\Delta_n^{3/2}}{u_n^{(r)} u_n^{(l)}}\right)$.

In view of (9), we get $T_1 \rightarrow^P 0$. \square

Proof of Theorem 2. To prove the theorem, it suffices to show that

$$JHY[Y^C] \rightarrow \int_0^T \sigma_{1s} \sigma_{2s} ds \quad \text{and} \quad JHY[Y] - JHY[Y^C] \rightarrow 0.$$

The first asymptotic follows from Christensen et al. (2010), and the second one can be proved by using the same procedure in Theorem 1. \square

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